Calculations and Discussion of the Examples in "U-Statistics and Imperfect Ranking in Ranked Set Sampling"

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1 Discussion

In light of misunderstandings by readers of earlier drafts of this paper, some discussion of issues in modeling judgement ranking seems to be needed. The main point to be made is that modeling the judgement ranking process by conditioning on the unmeasured data is not only correct, but is a natural and perhaps the only reasonable way to proceed. We believe that some reflection will convince the reader that this view is correct.

A useful thought experiment is to imagine judgement ranking a pair of books selected at random from the shelf (without checking the actual number of pages of course). If the two books selected happen to be very different in length, then they are easy to rank correctly. However if the numbers of pages of the two books are very nearly the same, it is very difficult to guess which is actually larger. The crucial point is that it is the actual (ordered) lengths of the books, $X_{(1)}, X_{(2)}$, which largely determine the difficulty of ranking, and hence the *conditional* probability that we get it right. Modeling judgement ranking using only the expected values of $X_{(1)}, X_{(2)}$, for example, does not capture this aspect of judgement ranking at all, since seen through the expected values all samples look the same. Expected values can at most tell us something about the average difficulty of ranking over all possible samples, and are thus not a suitable basis for modeling the ranking process itself.

The remainder of this note simply provides detailed calculations for the examples that were used in the paper to illustrate some errors in the literature. This is all elementary, and there's no doubt that much of it could be done more elegantly. The objective is simply to demonstrate the calculations in sufficient detail that they can be seen to be correct with minimal effort on the part of the reader.

2 The First Example

Suppose that $k = 2$, and that $X_1, X_2 \sim$ i.i.d. $U(0,1)$. We model the judgement ranking scheme by assuming that conditional on X_1, X_2,

$$J_1 = 1 \quad \text{with probability} \quad (1/2)\big[1 + (X_{(2)} - X_{(1)})\big]\,. \tag{1}$$

Note that $P(J_1 = 1|X_1, X_2)$ is close to 1/2 if $X_{(2)} - X_{(1)}$ is small, and is close to 1 if $X_{(2)} - X_{(1)}$ is close to 1, the maximum difference in this example.

The actual value of $F_{[1]}(1/2)$ can be calculated as follows:

$$\begin{aligned}
F_{[1]}(1/2) &= P(X_{[1]} \le 1/2) \\
&= P(X_{[1]} \le 1/2, X_1 \le 1/2, X_2 \le 1/2) + P(X_{[1]} \le 1/2, X_1 > 1/2, X_2 > 1/2) \\
&\quad\; + P(X_{[1]} \le 1/2, X_1 \le 1/2, X_2 > 1/2) + P(X_{[1]} \le 1/2, X_1 > 1/2, X_2 \le 1/2) \\
&= P(X_1 \le 1/2, X_2 \le 1/2) + 0 \\
&\quad\; + P(X_{[1]} \le 1/2, X_1 \le 1/2, X_2 > 1/2) + P(X_{[1]} \le 1/2, X_1 > 1/2, X_2 \le 1/2) \\
&= \frac{1}{4} + 2P(X_{[1]} \le 1/2, X_1 \le 1/2, X_2 > 1/2) \\
&= \frac{1}{4} + 2P(J_1 = 1, X_1 \le 1/2, X_2 > 1/2) \\
&= \frac{1}{4} + 2\int_{1/2}^{1}\int_{0}^{1/2} P(J_1 = 1|X_1 = x_1, X_2 = x_2)\, dx_1\, dx_2 \\
&= \frac{1}{4} + 2\int_{1/2}^{1}\int_{0}^{1/2} \frac{1}{2}[1 + x_2 - x_1]\, dx_1\, dx_2 = \frac{5}{8}\,.
\end{aligned}$$

But

$$\begin{aligned}
p_{11} = P(J_1 = 1) &= E\{P(J_1 = 1|X_1, X_2)\} = E\{(1/2)[1 + (X_{(2)} - X_{(1)})]\} \\
&= (1/2)[1 + ((2/3) - (1/3))] = 2/3\,,
\end{aligned}$$

so that $p_{11} = p_{22} = 1 - p_{12} = 1 - p_{21} = 2/3$. Of course $F_{(1)}(1/2) = 3/4$ and $F_{(2)}(1/2) = 1/4$, so that substitution into equation (10) of the paper yields

$$p_{11}F_{(1)}(1/2) + p_{21}F_{(2)}(1/2) = \frac{2}{3}\cdot\frac{3}{4} + \frac{1}{3}\cdot\frac{1}{4} = \frac{7}{12}\,.$$

This proves that equation (10) does not yield the correct value of $F_{[1]}(1/2)$.

3 The Example with Normal Errors

Suppose that $k = 2$, and that $X_1, X_2 \sim$ i.i.d. $N(0,1)$. We model the judgement ranking scheme by assuming that the observer's perception of X_1, X_2 is clouded by noise, i.e., that the judgement ranks assigned to X_1 and X_2 are actually the joint ranks of T_1 and T_2, where $T_i = X_i + \epsilon_i$, $i = 1, 2$, with $\epsilon_1, \epsilon_2 \sim$ i.i.d. $N(0, \sigma^2)$, independently of X_1, X_2. As in the previous example it is clear that if X_1 and X_2 are close together (relative to σ) then the conditional probability of correct ranking will be approximately 1/2, while if if X_1 and X_2 are well separated, then the conditional probability of correct ranking will be large. This model was used in a simulation study by Dell and Clutter (1972).

Note that

$$\begin{aligned} P(J_1 = 1) &= P(X_1 \le X_2, T_1 \le T_2) + P(X_1 > X_2, T_1 > T_2) \\ &= 2P(X_1 \le X_2, T_1 \le T_2)\,. \end{aligned}$$

Let ϕ and Φ represent the standard normal density and distribution functions, respectively. Then

$$\begin{aligned} &P(X_1 \le X_2, T_1 \le T_2) \\ &\quad = P(X_1 - X_2 \le 0, X_1 - X_2 \le \epsilon_2 - \epsilon_1) \\ &\quad = P(X_1 - X_2 \le 0, X_1 - X_2 \le \epsilon_2 - \epsilon_1, \epsilon_2 - \epsilon_1 > 0) \\ &\qquad\quad + P(X_1 - X_2 \le 0, X_1 - X_2 \le \epsilon_2 - \epsilon_1, \epsilon_2 - \epsilon_1 \le 0) \\ &\quad = P(X_1 - X_2 \le 0, \epsilon_2 - \epsilon_1 > 0) + P(X_1 - X_2 \le \epsilon_2 - \epsilon_1 \le 0) \\ &\quad = \frac{1}{4} + \int_{-\infty}^{0}\int_{-\infty}^{w_2} \frac{1}{\sqrt{2}}\,\phi(w_1/\sqrt{2})\,\frac{1}{\sqrt{2}\sigma}\,\phi(w_2/(\sqrt{2}\sigma)\,dw_1\,dw_2\,, \end{aligned}$$

since $X_1 - X_2 \sim N(0,2)$ and $\epsilon_2 - \epsilon_1 \sim N(0, 2\sigma^2)$ are independent. With the change of variables $z_1 = w_1/\sqrt{2}$ and $z_2 = w_2/(\sqrt{2}\sigma)$, this becomes

$$\begin{aligned} P(X_1 \le X_2, T_1 \le T_2) &= \frac{1}{4} + \int_{-\infty}^{0} \int_{-\infty}^{\sigma z_2} \phi(z_1)\, \phi(z_2)\, dz_1\, dz_2 \\ &= \frac{1}{2} - \frac{1}{2\pi} \arctan(\sigma)\,. \end{aligned}$$

The last equality follows from purely geometric considerations after noting that the bivariate density function $\phi(z_1)\,\phi(z_2)$ is spherically symmetric about the origin. Thus we have

$$p_{11} = P(J_1 = 1) = 1 - \frac{1}{\pi} \arctan(\sigma)\,,$$

so that

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 1 - \arctan(\sigma)/\pi & \arctan(\sigma)/\pi \\ \arctan(\sigma)/\pi & 1 - \arctan(\sigma)/\pi \end{bmatrix}. \tag{2}$$

Note that as $\sigma \to 0$ (perfect ranking) this converges to the identity matrix, while as $\sigma \to \infty$, in which case the noise becomes so great that we are effectively choosing the judgement ranks completely at random, it converges to a matrix with all entries equal to $1/2$.

Considering $F_{[1]}$ and $F_{[2]}$, note that

$$\begin{aligned} F_{[1]}(x) &= P(X_1 \le x, T_1 \le T_2) + P(X_2 \le x, T_1 > T_2) \\ &= 2P(X_1 \le x, T_1 \le T_2)\,, \end{aligned}$$

and

$$\begin{aligned} F_{[2]}(x) &= P(X_1 \le x, T_1 > T_2) + P(X_2 \le x, T_1 \le T_2) \\ &= 2P(X_1 \le x, T_1 > T_2) \\ &= 2\big[P(X_1 \le x) - P(X_1 \le x, T_1 \le T_2)\big]\,. \end{aligned}$$

A simple conditioning argument shows that

$$
\begin{aligned}
P(X_1 \le x, T_1 \le T_2) &= P(X_1 \le x, X_2 + \epsilon_2 - \epsilon_1 \ge X_1) \\
&= \int_{-\infty}^{x} P(X_2 + \epsilon_2 - \epsilon_1 \ge X_1 | X_1 = z)\, \phi(z)\, dz \\
&= \int_{-\infty}^{x} P(X_2 + \epsilon_2 - \epsilon_1 \ge z)\, \phi(z)\, dz \\
&= \int_{-\infty}^{x} P\left(\frac{X_2 + \epsilon_2 - \epsilon_1}{\sqrt{1+2\sigma^2}} \ge \frac{z}{\sqrt{1+2\sigma^2}} \right) \phi(z)\, dz \\
&= \int_{-\infty}^{x} \big[1 - \Phi\big(z/\sqrt{1+2\sigma^2}\big)\big] \phi(z)\, dz\,,
\end{aligned}
$$

and this in turn implies that $X_{[1]}$ and $X_{[2]}$ have distribution functions

$$
F_{[1]}(x) = 2\int_{-\infty}^{x} \big[1 - \Phi\big(z/\sqrt{1+2\sigma^2}\big)\big] \phi(z)\, dz
$$

and

$$
F_{[2]}(x) = 2\int_{-\infty}^{x} \Phi\big(z/\sqrt{1+2\sigma^2}\big) \phi(z)\, dz\,,
$$

respectively. The corresponding densities are thus

$$
f_{[1]}(x) = 2\big[1 - \Phi\big(x/\sqrt{1+2\sigma^2}\big)\big] \phi(x)
$$

and

$$
f_{[2]}(x) = 2\Phi\big(x/\sqrt{1+2\sigma^2}\big) \phi(x)\,,
$$

Note that $F_{[1]}(x) + F_{[2]}(x) = 2\Phi(x)$, which we know must hold. Of course the distribution functions of $X_{(1)}$ and $X_{(2)}$ are given by

$$
F_{(1)}(x) = 1 - \big[1 - \Phi(x)\big]^2
$$

and

$$
F_{(2)}(x) = \big[\Phi(x)\big]^2\,,
$$

respectively, and it is clear at this point that, e.g., $F_{[1]}(x) \neq p_{11}F_{(1)}(x) + p_{21}F_{(2)}(x)$, as claimed elsewhere.

For explicit verification of this in a specific case, consider $x = 0$. Here

$$\begin{aligned} F_{[2]}(0) &= 2\int_{-\infty}^{0} \Phi(z/\sqrt{1+2\sigma^2})\phi(z)\,dz \\ &= 2\int_{-\infty}^{0}\int_{-\infty}^{z/\sqrt{1+2\sigma^2}} \phi(y)\phi(z)\,dy\,dz \\ &= \frac{1}{\pi}\arctan(\sqrt{1+2\sigma^2})\,, \end{aligned}$$

where again the last equality follows from geometric considerations and the fact that $\phi(y)\phi(z)$ is a spherically symmetric bivariate density function. Of course

$$F_{[1]}(0) = 2\Phi(0) - F_{[2]}(0) = 1 - \frac{1}{\pi}\arctan(\sqrt{1+2\sigma^2})\,.$$

But

$$\begin{aligned} p_{11}F_{(1)}(0) + p_{21}F_{(2)}(0) &= \frac{3}{4}\Big(1 - \frac{1}{\pi}\arctan(\sigma)\Big) + \frac{1}{4}\Big(\frac{1}{\pi}\arctan(\sigma)\Big) \\ &= \frac{3}{4} - \frac{1}{2\pi}\arctan(\sigma)\,, \end{aligned}$$

and this does agree with the correct expression for $F_{[1]}(0)$ for any $0 < \sigma < \infty$. For example, if $\sigma = 1$, then $F_{[1]}(0) = 2/3$, while the incorrect expression yields 5/8. Note however that the two expressions do agree when $\sigma = 0$ (perfect ranking) and in the limit as $\sigma \to \infty$ (completely random judgement ranking).

4 The Sign Test

4.1 Applied to the First Example

Hettmansperger's equation (5) is

$$\delta_p^2 = 1 - \frac{4}{k}\sum_{j=1}^{k}\Big\{\sum_{i=1}^{k} p_{ij}\Big(F_{(i)}(0) - \frac{1}{2}\Big)\Big\}^2\,, \tag{H5}$$

while the correct expression is

$$\delta_p^2 = 1 - \frac{4}{k}\sum_{j=1}^{k}\Big\{F_{[j]}(0) - \frac{1}{2}\Big\}^2\,. \tag{H5$'$}$$

To see that these expression are not equivalent, we apply both to the first example (replacing $F_{(i)}(0)$ and $F_{[i]}(0)$ by $F_{(i)}(1/2)$ and $F_{[i]}(1/2)$, respectively, to reflect the fact that the median of the $U(0,1)$ distribution is $1/2$.

We have already established that $F_{[1]}(1/2) = 5/8$. By similar calculations, or merely from symmetry considerations, $F_{[2]}(1/2) = 3/8$. Since $k = 2$ in this example, substitution into the correct expression (H5′) yields

$$\delta_p^2 = 1 - 2\Big\{\Big(\frac{5}{8} - \frac{1}{2}\Big)^2 + \Big(\frac{3}{8} - \frac{1}{2}\Big)^2\Big\} = \frac{15}{16},$$

as claimed in the paper.

On the other hand, since $F_{(1)}(1/2) = 3/4$ and $F_{(2)}(1/2) = 1/4$, and $p_{11} = p_{22} = 1 - p_{12} = 1 - p_{21} = 2/3$, substitution into the incorrect expression (H5) yields $35/36$.

4.2 Applied to the Normal Errors Example

Following the notation in the paper, the correct value for Hettmansperger's δ_p in the normal errors example is

$$\begin{aligned}\delta_p^2 &= 1 - 2\Big\{\big(F_{[1]}(0) - \frac{1}{2}\big)^2 + \big(F_{[2]}(0) - \frac{1}{2}\big)^2\Big\}\\ &= 1 - 4\Big[\frac{1}{\pi}\arctan(\sqrt{1+2\sigma^2}) - \frac{1}{2}\Big]^2\end{aligned}$$

Hettmansperger's expression (5) on the other hand yields

$$\begin{aligned}\delta_p^2 = 1 - 2\Big\{&\Big[p_{11}\big(F_{(1)}(0) - \frac{1}{2}\big) + p_{21}\big(F_{(2)}(0) - \frac{1}{2}\big)\Big]^2\\ &+ \Big[p_{12}\big(F_{(1)}(0) - \frac{1}{2}\big) + p_{22}\big(F_{(2)}(0) - \frac{1}{2}\big)\Big]^2\Big\}\\ = 1 - 2\Big\{&\Big[\frac{1}{4}p_{11} - \frac{1}{4}p_{21}\Big]^2 + \Big[\frac{1}{4}p_{12} - \frac{1}{4}p_{22}\Big]^2\Big\}\\ = 1 - \frac{1}{8}\Big\{&\Big[p_{11} - p_{21}\Big]^2 + \Big[p_{12} - p_{22}\Big]^2\Big\}\\ = 1 - \frac{1}{4}&\Big[1 - \frac{2}{\pi}\arctan(\sigma)\Big]^2,\end{aligned}$$

and again this does not agree with the correct answer for any $0 < \sigma < \infty$ (though again they do agree at $\sigma = 0$ and ∞). For example, if $\sigma = 1$ the correct value for δ_p is $8/9$, while the incorrect expression yields $15/16$. In fact the incorrect result is exceeds the correct one for all $-\infty < \sigma < \infty$.

5 The Mann-Whitney Test Statistic

Recall that for the Mann-Whitney statistic, $h(x, y) = I(x < y)$, so that $h_{10}(x) = 1 - G(x)$ and $h_{01}(y) = F(y)$, where we assume F and G continuous. Thus, under ranked set sampling the asymptotic variance of U is given by Theorem 2 of the paper, with $\gamma_{r\cdot} = 1 - E[G(X_{[r]})]$ and $\gamma_{\cdot s} = E[F(Y_{[s]})]$. When $F = G$, the usual null hypothesis, $\theta = 1/2$, and $\zeta_{10} = \zeta_{01} = 1/12$.

If equation (10) of the paper holds, then

$$\begin{aligned}\gamma_{r\cdot} &= 1 - \int_{-\infty}^{\infty} G(x)\, dF_{[r]}(x) = 1 - \sum_{i=1}^{k} p_{ir} \int_{-\infty}^{\infty} G(x)\, dF_{(i)}(x) \\ &= 1 - \sum_{i=1}^{k} p_{ir} E[G(X_{(i)})]\,,\end{aligned}$$

and similarly $\gamma_{\cdot s} = \sum_{i=1}^{l} q_{is} E[F(Y_{(i)})]$, where the q_{ij} are defined in the same way as the p_{ij} but for the second sample. If $F = G$, the usual null hypothesis, then $G(X_{(i)})$ is distributed like the ith order statistic from a sample of k i.i.d. $U(0, 1)$ random variables, and thus $E[G(X_{(i)})] = i/(k + 1)$. Similarly, $E[F(Y_{(i)})] = i/(l + 1)$. Thus under the null hypothesis we have

$$\gamma_{r\cdot} = 1 - \frac{1}{k+1} \sum_{i=1}^{k} i p_{ir} \tag{3}$$

$$\gamma_{\cdot s} = \frac{1}{l+1} \sum_{i=1}^{l} i q_{is}\,. \tag{4}$$

Of course none of the results of this paragraph can be expected to hold for general judgement ranking schemes, but we note that after accomodating differences in notation, they provide simple alternatives to the more complicated expressions of Bohn and Wolfe (1994). In case ranking is perfect, so that the matrices of p_{ij}s and q_{ij}s are identity matrices, these reduce to $\gamma_{r\cdot} = 1 - r/(k+1)$ and $\gamma_{\cdot s} = s/(l+1)$, providing simple alternatives to the more complicated expression in Bohn and Wolfe (1992).

5.1 Applied to the First Example

Now assume that both samples are drawn form $U(0,1)$ populations with the judgement ranking scheme as described in the first example. Then

$$
\begin{aligned}
\gamma_{1\cdot} &= 1 - E(X_{[1]}) = 1 - E(X_{(J_1)}) = 1 - E\{E(X_{(J_1)}|X_1, X_2)\} \\
&= 1 - E\{X_{(1)}P(J_1 = 1|X_1, X_2) + X_{(2)}P(J_1 = 2|X_1, X_2)\} \\
&= 1 - E\{(1/2)[1 + (X_{(2)} - X_{(1)})]X_{(1)} + (1/2)[1 - (X_{(2)} - X_{(1)})]X_{(2)}\} \\
&= 1 - (1/2)E\{X_{(1)} + X_{(2)} - (X_{(2)} - X_{(1)})^2\} \\
&= 1 - (1/2)(1/3 + 2/3 - 1/6) = 7/12\,,
\end{aligned}
$$

where we have used the fact that

$$E\{(X_{(2)} - X_{(1)})^2\} = \int_0^1 \int_0^{x_1} (x_2 - x_1)^2 \, dx_1 \, dx_2 = 1/6\,.$$

Similar calculations show that

$$
\begin{aligned}
\gamma_{1\cdot} &= 1 - E(X_{[1]}) = 7/12 & \qquad \gamma_{2\cdot} &= 1 - E(X_{[2]}) = 5/12 \\
\gamma_{\cdot 1} &= E(Y_{[1]}) = 5/12 & \qquad \gamma_{\cdot 2} &= E(Y_{[2]}) = 7/12
\end{aligned}
$$

Thus

$$2\zeta_{10}^* = 2\zeta_{01}^* = \frac{1}{12} - \frac{1}{2}\left\{\left(\frac{7}{12} - \frac{1}{2}\right)^2 + \left(\frac{5}{12} - \frac{1}{2}\right)^2\right\} = \frac{11}{144}\,.$$

On the other hand, suppose that we naively assume that equation (10) of the paper holds, and thus proceed to apply formulas (3) and (4) above. Since $p_{11} = p_{22} = 1 - p_{12} = 1 - p_{21} = 2/3$, we then obtain

$$\gamma_{1\cdot} = 1 - \frac{1}{3}\left((1)\left(\frac{2}{3}\right) + (2)\left(\frac{1}{3}\right)\right) = \frac{5}{9}\,.$$

Similar calculations show that $\gamma_{1\cdot} = \gamma_{\cdot 2} = 5/9$ and $\gamma_{2\cdot} = \gamma_{\cdot 1} = 4/9$, so that

$$2\zeta_{10}^* = 2\zeta_{01}^* = \frac{1}{12} - \frac{1}{2}\left\{\left(\frac{5}{9} - \frac{1}{2}\right)^2 + \left(\frac{4}{9} - \frac{1}{2}\right)^2\right\} = \frac{13}{162}\,.$$

5.2 Applied to the Normal Errors Example

Assume that both samples, populations, and judgement ranking schemes follow the setup of the normal errors example: both populations follow a standard normal distribution, so that $F = G = \Phi$; $k = l = 2$; and the judgement ranking process for both samples is clouded by Gaussian noise with the same variance, σ^2. Then

$$\begin{aligned}
\gamma_{1\cdot} &= E[1 - \Phi(X_{[1]})] = \int_{-\infty}^{\infty} [1 - \Phi(x)] f_{[1]}(x)\, dx \\
&= 2 \int_{-\infty}^{\infty} [1 - \Phi(x)] \big[1 - \Phi\big(x/\sqrt{1+2\sigma^2}\big)\big] \phi(x)\, dx \\
&= 2E\big\{[1 - \Phi(Z)] \big[1 - \Phi\big(Z/\sqrt{1+2\sigma^2}\big)\big]\big\}\,,
\end{aligned}$$

where $Z \sim N(0,1)$. Either by noting that both $\Phi(Z)$ and $\Phi(Z/\sqrt{1+2\sigma^2})$ have expectation $1/2$, or by using the fact the Z is equal in distribution to $-Z$, it is easily seen that this in turn can be written more simply as

$$\gamma_{1\cdot} = 2E\big\{\Phi(Z)\Phi\big(Z/\sqrt{1+2\sigma^2}\big)\big\}\,.$$

Similarly we find that

$$\gamma_{1\cdot} = 1 - \gamma_{2\cdot} = 1 - \gamma_{\cdot 1} = \gamma_{\cdot 2} = 2E\big\{\Phi(Z)\Phi\big(Z/\sqrt{1+2\sigma^2}\big)\big\}\,,$$

and thus the asymptotic variance is correctly calculated using

$$\begin{aligned}
2\zeta_{10}^* = 2\zeta_{01}^* &= \frac{1}{12} - \frac{1}{2}\left[\left(\gamma_{1\cdot} - \frac{1}{2}\right)^2 + \left(1 - \gamma_{1\cdot} - \frac{1}{2}\right)^2\right] = \frac{1}{12} - \left(\gamma_{1\cdot} - \frac{1}{2}\right)^2 \\
&= \frac{1}{12} - \left(2E\big\{\Phi(Z)\Phi\big(Z/\sqrt{1+2\sigma^2}\big)\big\} - \frac{1}{2}\right)^2.
\end{aligned}$$

Alternatively, suppose that we naively assume that equation (10) of the paper holds and apply (3) and (4). Then since $p_{11} = p_{22} = 1 - p_{12} = 1 - p_{21} = 1 - \arctan(\sigma)/\pi$, we obtain

$$\begin{aligned}
\gamma_{1\cdot} &= 1 - \frac{1}{3}\left((1)\left(1 - \frac{1}{\pi}\arctan(\sigma)\right) + (2)\left(\frac{1}{\pi}\arctan(\sigma)\right)\right) \\
&= \frac{2}{3} - \frac{1}{3\pi}\arctan(\sigma)\,.
\end{aligned}$$

Similar calculations show that

$$\gamma_{1\cdot} = 1 - \gamma_{2\cdot} = 1 - \gamma_{\cdot 1} = \gamma_{\cdot 2} = \frac{2}{3} - \frac{1}{3\pi}\arctan(\sigma)\,,$$

so that

$$\begin{aligned}
2\zeta_{10}^{*} = 2\zeta_{01}^{*} &= \frac{1}{12} - \frac{1}{2}\left[\left(\gamma_{1\cdot} - \frac{1}{2}\right)^2 + \left(1 - \gamma_{1\cdot} - \frac{1}{2}\right)^2\right] = \frac{1}{12} - \left(\gamma_{1\cdot} - \frac{1}{2}\right)^2 \\
&= \frac{1}{12} - \left(\frac{1}{6} - \frac{1}{3\pi}\arctan(\sigma)\right)^2 \\
&= \frac{1}{12} - \frac{1}{9}\left(\frac{1}{2} - \frac{1}{\pi}\arctan(\sigma)\right)^2 .
\end{aligned}$$

Once again, although this agrees with the correct expression at $\sigma = 0$ and $\sigma = \infty$, they do not agree for any $0 < \sigma < \infty$. For example, if $\sigma = 0.5$, then numerical integration shows that the correct value of $2\zeta_{10}^{*} = 2\zeta_{01}^{*}$ is .0616, while the second, incorrect formula yields $11/144 \approx .0695$, which is about thirteen percent too large.

References

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